

CERTAIN PARANORMED FRACTIONAL ORDERED PASCAL DIFFERENCE SEQUENCE SPACES

Diptimayee Jena and Salila Dutta

Department of Mathematics,
Utkal University, Bhubaneswar, Odisha, INDIA

E-mail : jena.deeptimayee@gmail.com, saliladutta516@gmail.com

(Received: Oct. 11, 2024 Accepted: Dec. 26, 2024 Published: Dec. 30, 2024)

Abstract: In this article, we introduce certain new paranormed Pascal difference sequence spaces of fractional order τ . Some topological properties of these spaces are studied here. We determined α -, β - and γ - duals and characterized some matrix transformations of the spaces.

Keywords and Phrases: Sequence spaces, fractional difference operator Δ^{τ} , Schauder basis, Pascal mean, dual spaces.

2020 Mathematics Subject Classification: 46A45, 46A70, 46B03, 46B15.

1. Introduction

Suppose the set of all real or complex sequences is denoted by Λ . By ℓ_{∞} , c_0 and c , we denote the spaces of all bounded, null and convergent sequences, respectively, which are subspaces of Λ normed by $\|x\|_{\infty} = \sup_k |x_k|$. Also the spaces of all convergent, bounded and absolutely convergent series are denoted by cs , bs and ℓ_1 , respectively. Through out the paper summation without limits runs from 0 to ∞ . For $p = (p_k)$ a bounded sequence of strictly positive real numbers, Maddox [18] introduced $c_0(p)$, $c(p)$ and $\ell_{\infty}(p)$ as:

$$\begin{aligned} c_0(p) &= \left\{ \xi = (\xi_k) \in \Lambda : \lim_{k \rightarrow \infty} |\xi_k|^{p_k} = 0 \right\}, \\ c(p) &= \left\{ \xi = (\xi_k) \in \Lambda : \lim_{k \rightarrow \infty} |\xi_k - l|^{p_k} = 0 \text{ for some } l \in \mathbb{R} \right\}, \\ \ell_{\infty}(p) &= \left\{ \xi = (\xi_k) \in \Lambda : \sup_{k \in \mathbb{N}} |\xi_k|^{p_k} < \infty \right\} \end{aligned}$$

and these are complete spaces with the paranorm

$$g(\xi) = \sup_{k \in \mathbb{N}} |\xi_k|^{p_k/M}, \quad \text{where } M = \max\{1, \sup_k p_k\}.$$

Now, define the set $\mathcal{M}(X, Y) = \{\zeta = (\zeta_k) : \xi\zeta = (\xi_k\zeta_k) \in Y, \forall \xi \in X\}$. The α -, β - and γ -duals of a sequence space X are as follows:

$$X^\alpha = \mathcal{M}(X, \ell_1), \quad X^\beta = \mathcal{M}(X, cs) \text{ and } X^\gamma = \mathcal{M}(X, bs).$$

The difference sequence spaces $X(\Delta) = \{\xi = (\xi_k) : \Delta(\xi_k) \in X\}$ for $X = \{c_0, c, \ell_\infty\}$ was introduced by Kizmaz [17], which was generalized by Et and Çolak [12]. Then difference sequence spaces attracted the attention of several authors [5, 12] in different directions. Recently, Jena and Dutta [[14], [15]] introduced Bell difference sequence spaces and fractional ordered Euler Riesz difference sequence spaces. The difference space bv_p consisting of the sequence $x = (x_k)$ such that $(x_k - x_{k-1}) \in \ell_p$ have been introduced in the case $0 < p \leq 1$ by Altay and Başar [1], and in the case $1 \leq p \leq \infty$ by Başar and Altay [10]. The reader also refer to the recent monographs [9] and [20], and references therein, devoted to the matrix transformations and related topics, and the papers [2], [3], [6], [19], [22] and [23] on the domain of certain triangles in some paranormed Maddox sequence spaces.

For a proper fraction τ , Baliarsingh [7], Baliarsingh & Dutta [8] introduced the fractional difference operator $\Delta^{(\tau)}$ as

$$\Delta^{(\tau)}x_k = \sum_j (-1)^j \frac{\Gamma(\tau+1)}{j!\Gamma(\tau-j+1)} x_{k-j}, \quad (1.1)$$

along with its inverse

$$\Delta^{(-\tau)}x_k = \sum_j (-1)^j \frac{\Gamma(-\tau+1)}{j!\Gamma(-\tau-j+1)} x_{k-j}. \quad (1.2)$$

It is assumed that the series of fractional difference operators are convergent.

The Pascal mean P described as Pascal matrix $P = p_{mk}$ is defined by (see [21]).

$$P = p_{mk} = \begin{cases} \binom{m}{m-k} & \text{if } 0 \leq k \leq m \\ 0 & \text{if } k > m. \end{cases}$$

Its inverse P^{-1} is given by

$$(P^{-1})_{mk} = \begin{cases} (-1)^{m-k} \binom{m}{m-k} & \text{if } 0 \leq k \leq m \\ 0 & \text{if } k > m. \end{cases} \quad (1.3)$$

Pascal sequence spaces p_∞ , p_0 and p_c are introduced by Polat [21]. Then Aydın and Polat [5] introduced Pascal difference sequence spaces of integer order m as,

$$X(\Delta^{(m)}) = \{\xi = (\xi_k) \in \Lambda : \Delta^{(m)}\xi \in X\}, \quad (1.4)$$

for $X = \{p_\infty, p_0, p_c\}$.

Recently, Dutta and Jena [11] introduced the matrix $\tilde{\mathfrak{P}} = (P\Delta^{(\tau)})$ by combining Pascal mean P and fractional difference operator $\Delta^{(\tau)}$, which can be expressed as

$$\tilde{\mathfrak{P}} = (P\Delta^{(\tau)})_{mk} = \begin{cases} \sum_{l=k}^m (-1)^{l-k} \binom{m}{m-l} \frac{\Gamma(\tau+1)}{(l-k)!\Gamma(\tau-l+k+1)}, & \text{if } 0 \leq k \leq m \\ 0, & \text{if } k > m. \end{cases} \quad (1.5)$$

Equivalently, it is written as

$$\tilde{\mathfrak{P}}^{\tilde{\tau}} = (\mathfrak{p}_{mk}) = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ (1 - \tilde{\tau}) & 1 & 0 & 0 & \dots \\ (1 - 2\tilde{\tau} + \frac{\tilde{\tau}(\tilde{\tau}-1)}{2!}) & (2 - \tilde{\tau}) & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

and the inverse of $\tilde{\mathfrak{P}} = (P(\Delta^{(\tau)}))$ is given by

$$(\tilde{\mathfrak{P}}^{-1})_{mk} = \begin{cases} \sum_{j=k}^m (-1)^{m-k} \binom{j}{j-k} \frac{\Gamma(-\tau+1)}{(m-j)!\Gamma(-\tau-m+j+1)}, & \text{if } 0 \leq k \leq m \\ 0, & \text{if } k > m. \end{cases}$$

Now, we have

$$\begin{aligned} (P(\Delta^{(\tau)})x)_n &= \sum_{k=0}^n \binom{n}{n-k} \sum_{l=0}^n (-1)^l \frac{\Gamma(\tau+1)}{l!\Gamma(\tau-l+1)} x_{k-l} \\ &= \sum_{k=0}^n \left[\sum_{l=k}^n (-1)^{l-k} \binom{n}{n-l} \frac{\Gamma(\tau+1)}{(l-k)!\Gamma(\tau-l+k+1)} \right] x_k, \end{aligned}$$

for $n, k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. In particular for $n = 0$, we have $(P(\Delta^{(\tau)})x)_0 = x_0$, and similarly, $(P(\Delta^{(\tau)})x)_1 = (1 - \tau)x_0 + x_1$,

$$(P(\Delta^{(\tau)})x)_2 = \left(1 - 2\tau + \frac{\tau(\tau-1)}{2}\right)x_0 + (2 - \tau)x_1 + x_2, \dots \text{ and so on.}$$

Lemma 1.1. [11] *The operator $\tilde{\mathfrak{P}} = P(\Delta^{(\tau)})$ is linear.*

Now our interest is to introduce the new paranormed difference sequence spaces of fractional order which generalizes many known spaces. We introduce the spaces $c_0(\tilde{\mathfrak{P}}, p)$, $c(\tilde{\mathfrak{P}}, p)$, $\ell_\infty(\tilde{\mathfrak{P}}, p)$ by using the product of Pascal mean P with fractional operator $\Delta^{(\tau)}$. We prove certain topological properties and characterize the matrix transformations of these spaces.

2. New sequence spaces

Now, for a positive real number τ and a bounded sequence $p = (p_k)$, we define the following classes

$$\begin{aligned} c_0(\tilde{\mathfrak{P}}, p) &= \left\{ \xi = (\xi_k) \in \Lambda : \tilde{\mathfrak{P}}\xi \in c_0(p) \right\}, \\ \ell_\infty(\tilde{\mathfrak{P}}, p) &= \left\{ \xi = (\xi_k) \in \Lambda : \tilde{\mathfrak{P}}\xi \in \ell_\infty(p) \right\}, \\ c(\tilde{\mathfrak{P}}, p) &= \left\{ \xi = (\xi_k) \in \Lambda : \tilde{\mathfrak{P}}\xi \in c(p) \right\}. \end{aligned}$$

The above classes generalize the following spaces as follows:

(i) For $\tau = 0$ and $p = e = (1, 1, 1, \dots)$, it reduces to p_c, p_0 and p_∞ , studied by Polat [21].

(ii) For $\tau = 1$ and $p = e$, it reduces to $p_c(\Delta), p_0(\Delta), p_\infty(\Delta)$, studied by Aydın and Polat [4].

(iii) For $\tau = m$ and $p = e$, it reduces to $p_c(\Delta^{(m)}), p_0(\Delta^{(m)})$ and $p_\infty(\Delta^{(m)})$, where $\Delta^{(m)} = \sum_{i=0}^m -1^j \binom{m}{j} x_{m-j}$ studied by Aydın and Polat [5].

(iv) For $p = e = (1, 1, 1, \dots)$, it reduces to $\ell_\infty(\tilde{\mathcal{P}}^{\tilde{\tau}})$, introduced by Dutta and Jena [11].

Now with $\tilde{\mathfrak{P}}$ - transform of $\xi = (\xi_n)$ we define the sequence $\zeta = (\zeta_n)$, $(n \in \mathbb{N})$ as follows :

$$\zeta_n = \left(\tilde{\mathfrak{P}}\xi \right)_n = \sum_{k=0}^n \sum_{l=k}^n (-1)^{l-k} \binom{n}{n-l} \frac{\Gamma(\tau+1)}{(i-k)! \Gamma(\tau-l+k+1)} \xi_k. \quad (2.1)$$

Theorem 2.1. *The spaces $\ell_\infty(\tilde{\mathfrak{P}}, p)$, $c_0(\tilde{\mathfrak{P}}, p)$ and $c(\tilde{\mathfrak{P}}, p)$ are linear spaces paranormed by*

$$g_1(\xi) = \sup_n \left| \sum_{k=0}^n \sum_{l=k}^n (-1)^{l-k} \binom{n}{n-l} \frac{\Gamma(\tau+1)}{(i-k)! \Gamma(\tau-l+k+1)} \xi_k \right|^{\frac{p_k}{M}} \quad (2.2)$$

if and only if $\inf_k p_k > 0$.

Proof. Consider the space $c_0(\tilde{\mathfrak{P}}, p)$. We observe that $g_1(\theta) = \theta$, where $\theta = (0, 0, 0, 0, \dots)$ and $g_1(-\xi) = g_1(\xi)$ for all $\xi \in c_0(\tilde{\mathfrak{P}}, p)$. For linearity of $c_0(\tilde{\mathfrak{P}}, p)$ with respect to coordinate wise addition and scalar multiplication, consider any two sequences ξ_1 and $\xi_2 \in c_0(\tilde{\mathfrak{P}}, p)$ and scalars $\alpha, \beta \in \mathbb{R}$. Since the operator $\Delta^{(\tau)}$

is linear [8],

$$\begin{aligned}
 & g_1(\alpha\xi_1 + \beta\xi_2) \\
 &= \sup_n \left| \sum_{k=0}^n \left(\sum_{l=k}^n (-1)^{l-k} \binom{n}{n-l} \frac{\Gamma(\tau+1)}{(i-k)!\Gamma(\tau-l+k+1)} \right) (\alpha\xi_{1_k} + \beta\xi_{2_k}) \right|^{\frac{p_k}{M}} \\
 &\leq \max\{1, |\alpha|\} \sup_n \left| \sum_{k=0}^n \left(\sum_{l=k}^n (-1)^{l-k} \binom{n}{n-l} \frac{\Gamma(\tau+1)}{(i-k)!\Gamma(\tau-l+k+1)} \right) \xi_{1_k} \right|^{\frac{p_k}{M}} \\
 &\quad + \max\{1, |\beta|\} \sup_n \left| \sum_{k=0}^n \left(\sum_{l=k}^n (-1)^{l-k} \binom{n}{n-l} \frac{\Gamma(\tau+1)}{(i-k)!\Gamma(\tau-l+k+1)} \right) \xi_{2_k} \right|^{\frac{p_k}{M}} \\
 &= \max\{1, |\alpha|\} g_1(\xi_1) + \max\{1, |\beta|\} g_1(\xi_2).
 \end{aligned}$$

So, g_1 is continuous with respect to scalar multiplication. The proof for other spaces follows with the similar techniques.

Theorem 2.2. *The spaces $\ell_\infty(\tilde{\mathfrak{P}}, p)$, $c_0(\tilde{\mathfrak{P}}, p)$ and $c(\tilde{\mathfrak{P}}, p)$ are complete linear spaces paranormed by $g_1(\xi)$, defined in inequality (2.2).*

Proof. Consider a Cauchy sequence $\{\xi^n\}$ in $\ell_\infty(\tilde{\mathfrak{P}}, p)$ and $\xi^n = \{\xi_0^{(n)}, \xi_1^{(n)}, \xi_2^{(n)}, \dots\}$. For given $\epsilon > 0$, $\exists$ a positive integer $N_0(\epsilon)$ such that $g_1(\xi^n - \xi^m) < \epsilon$, $\forall n, m \leq N_0(\epsilon)$. Then

$$\left| \left(\tilde{\mathfrak{P}}(\xi^n) \right)_k - \left(\tilde{\mathfrak{P}}(\xi^m) \right)_k \right|^{\frac{p_k}{M}} \leq \sup_k \left| \left(\tilde{\mathfrak{P}}(\xi^n) \right)_k - \left(\tilde{\mathfrak{P}}(\xi^m) \right)_k \right|^{\frac{p_k}{M}} < \epsilon,$$

where $\left\{ \left(\tilde{\mathfrak{P}}(\xi^0) \right)_k, \left(\tilde{\mathfrak{P}}(\xi^1) \right)_k, \dots \right\}$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, the sequence $\left(\tilde{\mathfrak{P}}(\xi^n) \right)_k$ converges and suppose that $\left(\tilde{\mathfrak{P}}(\xi^n) \right)_k \rightarrow (\xi)_k$ as $n \rightarrow \infty$. For each fixed $k \in \mathbb{N}_0$, $m \rightarrow \infty$ and $n \geq N_0(\epsilon)$, we have

$$\left| \left(\tilde{\mathfrak{P}}(\xi^n) \right)_k - \left(\tilde{\mathfrak{P}}(\xi) \right)_k \right|^{\frac{p_k}{M}} \leq \epsilon \tag{2.3}$$

But $\xi^n = \{\xi_k^{(n)}\} \in \ell_\infty(\tilde{\mathfrak{P}}, p)$, Therefore,

$$\left| \left(\tilde{\mathfrak{P}}(\xi^n) \right)_k \right|^{\frac{p_k}{M}} \leq \infty. \tag{2.4}$$

From inequalities (2.3) and (2.4) we have,

$$\left| \left(\tilde{\mathfrak{P}}\xi \right)_k \right|^{\frac{p_k}{M}} \leq \left| \left(\tilde{\mathfrak{P}}(\xi^n) \right)_k - \left(\tilde{\mathfrak{P}}\xi \right)_k \right|^{\frac{p_k}{M}} + \left| \left(\tilde{\mathfrak{P}}(\xi^n) \right)_k \right|^{\frac{p_k}{M}} \leq \infty.$$

This means that the sequence $\tilde{\mathfrak{P}}\xi \in \ell_\infty(\tilde{\mathfrak{P}}, p)$. This step completes the proof of the theorem.

Theorem 2.3. *The spaces $c_0(\tilde{\mathfrak{P}}, p)$, $c(\tilde{\mathfrak{P}}, p)$, $\ell_\infty(\tilde{\mathfrak{P}}, p)$ are linearly isomorphic to $c_0(p)$, $c(p)$, $\ell_\infty(p)$ where $0 < p_k \leq H < \infty$, respectively.*

Proof. Consider a linear mapping $T : \ell_\infty(\tilde{\mathfrak{P}}, p) \rightarrow \ell_\infty(p)$ defined by $\zeta = T\xi$.

Clearly $T(\xi) = \theta$ when $\xi = \theta$, so T is one-one. Let $\zeta \in \ell_\infty(p)$, define a sequence $\xi = (\xi_k)$ as

$$\xi_k = \sum_{j=0}^k \sum_{l=j}^k (-1)^{k-j} \binom{l}{l-j} \frac{\Gamma(-\tau+1)}{(k-l)!\Gamma(-\tau-k+l+1)} \zeta_j, (k \in \mathbb{N}) \quad (2.5)$$

Then

$$\begin{aligned} g_1(\xi) &= \sup_n \left| \sum_{k=0}^n \sum_{l=k}^n (-1)^{l-k} \binom{n}{n-l} \frac{\Gamma(\tau+1)}{(i-k)!\Gamma(\tau-l+k+1)} \xi_k \right|^{\frac{p_k}{M}} \\ &= \left(\sup_k \left| \sum_{j=0}^k \sum_{l=j}^k (-1)^{l-j} \binom{k}{k-l} \frac{\Gamma(\tau+1)}{(l-j)!\Gamma(\tau-l+j+1)} \xi_j \right| \right)^{\frac{p_k}{M}} \\ &= \left(\sup_k |\delta_{kj} \zeta_j| \right)^{\frac{p_k}{M}} = \left(\sup_k |\zeta_k| \right)^{\frac{p_k}{M}} < \infty, \\ &\text{where } \delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases} \end{aligned}$$

Thus, $\xi \in \ell_\infty(\tilde{\mathfrak{P}}, p)$ and T is onto and paranorm preserving. Hence $\ell_\infty(\tilde{\mathfrak{P}}, p) \cong \ell_\infty(p)$. In a similar manner, we can prove for other spaces.

3. Duals

This section deals with α -, β - and γ - duals of the new sequence spaces. We now state certain lemmas which will be useful to prove theorems of this section.

Lemma 3.1. [13] $H \in (\ell_\infty(p), \ell(q))$ iff

$$\sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K} h_{nk} B^{\frac{1}{p_k}} \right|^{q_n} < \infty \quad \text{for all integers } B > 1. \quad (3.1)$$

Lemma 3.2. [13] $H \in (\ell_\infty(p), c(q))$ iff

$$\sup_n \sum_k |h_{nk}| B^{\frac{1}{p_k}} < \infty, \quad (3.2)$$

and

$$\lim_{n \rightarrow \infty} \left(\sum_k |h_{nk} - h_k| B^{\frac{1}{p_k}} \right)^{q_n} = 0, \quad (3.3)$$

$\exists (\alpha_k) \in H, p_k > 0.$

Lemma 3.3. [13] $H \in (\ell_\infty(p), \ell_\infty(q))$ iff

$$\sup_{n \in \mathbb{N}} \left(\sum_k |h_{nk}| B^{\frac{1}{p_k}} \right)^{q_n} < \infty. \quad (3.4)$$

Where $\mathcal{F}$ is the collection of all finite subsets of $\mathbb{N}_0$ and $K \in \mathcal{F}$ and $q = (q_n)$ be a bounded sequence of strictly positive real numbers.

Theorem 3.4. The α -, β - and γ - duals of the space $\ell_\infty(\tilde{\mathfrak{P}}, p)$ are given by

$$\begin{aligned} \left\{ \ell_\infty(\tilde{\mathfrak{P}}, p) \right\}^\alpha &= D_1(p), \\ \left\{ \ell_\infty(\tilde{\mathfrak{P}}, p) \right\}^\beta &= D_2(p) \cap cs, \\ \left\{ \ell_\infty(\tilde{\mathfrak{P}}, p) \right\}^\gamma &= D_2(p), \end{aligned}$$

where

$$\begin{aligned} D_1(p) &= \bigcap_{M > 1} \left\{ a = (a_k) : \sup_{k \in \tau} \sum_n \left| \sum_{k \in K} \left[\sum_{j=0}^n \sum_{l=j}^n \binom{l}{l-j} \frac{\Gamma(-\tau+1)(-1)^{n-j}}{(n-l)! \Gamma(-\tau-n+l+1)} \right] a_k \right| \right. \\ &\quad \left. M^{\frac{1}{p_k}} < \infty \right\}, \\ D_2(p) &= \bigcap_{M > 1} \left\{ a = (a_k) : \sup_n \sum_k \left| \sum_{i=k}^n \left[\sum_{j=0}^i \sum_{l=j}^i \binom{l}{l-j} \frac{\Gamma(-\tau+1)(-1)^{i-j}}{(i-l)! \Gamma(-\tau-i+l+1)} \right] a_i \right| \right. \\ &\quad \left. M^{\frac{1}{p_k}} < \infty \right\}, \end{aligned}$$

Proof. Considering $x = (x_k) \in \ell_\infty(\tilde{\mathfrak{P}}, p)$ via the sequence $y \in \ell_\infty(p)$ as

$$x_m = \sum_{j=k}^m \sum_{j=k}^m (-1)^{m-k} \binom{j}{j-k} \frac{\Gamma(-\tau+1)}{(m-j)! \Gamma(-\tau-m+j+1)} y_k.$$

For all $n \in \mathbb{N}_0$ we have

$$\begin{aligned} x_m a_m &= \left[\sum_{k=0}^m \sum_{j=k}^m (-1)^{m-k} \binom{j}{j-k} \frac{\Gamma(-\tau+1)}{(m-j)! \Gamma(-\tau-m+j+1)} y_k \right] a_m \\ &= \sum_{k=0}^m \left[\sum_{j=k}^m (-1)^{m-k} \binom{j}{j-k} \frac{\Gamma(-\tau+1)}{(m-j)! \Gamma(-\tau-m+j+1)} a_m \right] y_k \\ &= (Uy)_m, \quad \text{for } m \in \mathbb{N}. \end{aligned}$$

where $\tilde{U}_m = (\tilde{u}_{mk})$ is defined as

$$\tilde{u}_{mk} = \begin{cases} \sum_{j=k}^m (-1)^{m-k} \binom{j}{j-k} \frac{\Gamma(-\tau+1)}{(m-j)! \Gamma(-\tau-m+j+1)} a_m, & m \geq k, \\ 0, & k > m. \end{cases}$$

By Lemma 3.1 we conclude that $\left\{ \ell_\infty(\tilde{\mathfrak{P}}, p) \right\}^\alpha = D_1(p)$. Now for beta- dual, we consider

$$\begin{aligned} \sum_{i=0}^m x_i a_i &= \sum_{i=0}^m \left[\sum_{k=0}^m \sum_{j=k}^m (-1)^{m-k} \binom{j}{j-k} \frac{\Gamma(-\tau+1)}{(m-j)! \Gamma(-\tau-m+j+1)} y_k \right] a_m \\ &= \sum_{k=0}^m \left[\sum_{i=0}^m \sum_{j=k}^m (-1)^{m-k} \binom{j}{j-k} \frac{\Gamma(-\tau+1)}{(m-j)! \Gamma(-\tau-m+j+1)} a_i \right] y_k \\ &= (Vy)_m, \quad (m \in \mathbb{N}_0), \end{aligned}$$

where the matrix $V = (v_{mk})$ is defined by

$$v_{mk} = \begin{cases} \sum_{i=k}^m \sum_{j=k}^m (-1)^{m-k} \binom{j}{j-k} \frac{\Gamma(-\tau+1)}{(m-j)! \Gamma(-\tau-m+j+1)} a_i, & m \geq k, \\ 0, & k > m. \end{cases}$$

Therefore we deduce that $ax = (a_m x_m) \in cs$ whenever $x = (x_k) \in \left\{ \ell_\infty(\tilde{\mathfrak{P}}, p) \right\}$ iff $Vy \in c$ as $y = (y_k) \in \ell_\infty(p)$. By using Lemma 3.1 with $q = q_m = 1$ we conclude

that $\left\{ \ell_\infty \left(\tilde{\mathfrak{P}}, p \right) \right\}^\beta = D_2(p) \cap cs$. Hence the theorem proved and the duals of other spaces can be obtained in a similar manner using Lemma 3.2 and Lemma 3.3.

4. Matrix Transformations

This section deals with the matrix mappings from $\left\{ \ell_\infty \left(\tilde{\mathfrak{P}}, p \right) \right\}$ into different known sequence spaces.

We characterize the classes $\left(\ell_\infty \left(\tilde{\mathfrak{P}}, p \right), \check{Y}(p) \right)$ where $\check{Y} = \{c_0, c, \ell_\infty, \ell_1\}$.

For brevity, we write

$$\begin{aligned} \hat{h}_{nk} &= \sum_{i=k}^{\infty} \sum_{j=k}^m (-1)^{m-k} \binom{j}{j-k} \frac{\Gamma(-\tau+1)}{(m-j)! \Gamma(-\tau-m+j+1)} h_{ni} \\ a_{nk} &= \sum_{i=k}^m \sum_{j=k}^m (-1)^{m-k} \binom{j}{j-k} \frac{\Gamma(-\tau+1)}{(m-j)! \Gamma(-\tau-m+j+1)} h_{ni} \end{aligned}$$

for each $n, k \in \mathbb{N}$. Here certain known results are given which are essential for next theorems.

Lemma 4.1. [16] *Let λ be an FK space. For a triangle U and its inverse V with μ arbitrary subset of $\tilde{\omega}$ we have $H = (h_{nk}) \in (\lambda_U : \mu)$ iff $E^{(n)} = (e^{(n)}_{mk}) \in (\lambda : c)$ for all $n \in \mathbb{N}$ and $E = (e_{nk}) \in (\lambda : \mu)$, where*

$$e^{(n)}_{mk} = \begin{cases} \sum_{i=k}^m h_{ni} v_{ik}, & (0 \leq k \leq m) \\ 0, & (k > m) \end{cases} \text{ and } e_{nk} = \sum_{i=k}^{\infty} h_{ni} v_{ik}$$

$\forall n, m, k \in \mathbb{N}$. Consider

$$\sup_m \sum_{i=k}^m |a_{nk}|^q < \infty, \quad (4.1)$$

$$\lim_{m \rightarrow \infty} a_{nk} = \hat{h}_{nk}, \quad (4.2)$$

$$\lim_{m \rightarrow \infty} \sum_{i=k}^m |a_{nk}|, \quad (4.3)$$

$$\sup_{m,k} |a_{nk}| < \infty, \quad (4.4)$$

$$\sup_n \sum_k \left| \hat{h}_{nk} \right|^q < \infty, \quad (4.5)$$

$$\sup_k \sum_n \left| \hat{h}_{nk} \right| < \infty, \quad (4.6)$$

$$\lim_{n \rightarrow \infty} \sum_k \hat{h}_{nk} = 0, \quad (4.7)$$

$$\lim_{n \rightarrow \infty} \hat{h}_{nk} = \hat{\alpha}_k, \quad (4.8)$$

$$\lim_{n \rightarrow \infty} \sum_k \left| \hat{h}_{nk} \right| = \sum_k \left| \hat{\alpha}_k \right|, \quad (4.9)$$

$$\sup_{N \in \mathcal{F}} \sum_k \left| \sum_{n \in N} \hat{h}_{nk} \right|^q < \infty, \quad (4.10)$$

$$\sup_{n,k} \left| \hat{h}_{nk} \right| < \infty, \quad (4.11)$$

$$\sup_{N, K \in \mathcal{F}} \left| \sum_{n \in N} \sum_{k \in K} \hat{h}_{nk} \right| < \infty. \quad (4.12)$$

Using Lemma 3.1 and Lemma 4.1 we state the following theorems.

Theorem 4.2.

- (i) $H = (h_{nk}) \in \left(\ell_1 \left(\tilde{\mathfrak{P}}, p \right), c_0(p) \right)$ iff (4.2), (4.4), (4.8) with $\hat{\alpha}_k = 0$ and (4.11) hold.
- (ii) $H = (h_{nk}) \in \left(\ell_1 \left(\tilde{\mathfrak{P}}, p \right), c(p) \right)$ iff (4.2), (4.4), (4.8) and (4.11) hold.
- (iii) $H = (h_{nk}) \in \left(\ell_1 \left(\tilde{\mathfrak{P}}, p \right), \ell_\infty(p) \right)$ iff (4.2), (4.4) and (4.11) hold.
- (iv) $H = (h_{nk}) \in \left(\ell_1 \left(\tilde{\mathfrak{P}}, p \right), \ell_1(p) \right)$ iff (4.2), (4.4) and (4.6) hold.

Theorem 4.3.

- (i) $H = (h_{nk}) \in \left(\ell_\infty \left(\tilde{\mathfrak{P}}, p \right), c_0(p) \right)$ iff (4.2), (4.3) and (4.7) hold.
- (ii) $H = (h_{nk}) \in \left(\ell_\infty \left(\tilde{\mathfrak{P}}, p \right), c(p) \right)$ iff (4.2), (4.3), (4.8) and (4.9) hold.
- (iii) $H = (h_{nk}) \in \left(\ell_\infty \left(\tilde{\mathfrak{P}}, p \right), \ell_\infty(p) \right)$ iff (4.2), (4.3) and (4.5) with $q = 1$ hold.

(iv) $H = (h_{nk}) \in \left(\ell_\infty \left(\tilde{\mathfrak{P}}, p \right), \ell_1(p) \right)$ iff (4.2), (4.3) and (4.12) hold.

Acknowledgment

The authors would like to thank the anonymous referees for their careful reading and necessary comments which have improved the presentation of the paper.

References

- [1] Altay B., Başar F., The matrix domain and the fine spectrum of the difference operator Δ on the sequence space ℓ_p , $(0 < p < 1)$, Commun. Math. Anal., 2(2) (2007), 1–11.
- [2] Altay B., Başar F., Some paranormed sequence spaces of non-absolute type derived by weighted mean, J. Math. Anal. Appl., 319(2) (2006), 494–508.
- [3] Altay B., Başar F., Some paranormed Riesz sequence spaces of non-absolute type, Southeast Asian Bull. Math., 30(5) (2006), 591–608.
- [4] Aydın S., Polat H., Difference sequence spaces derived by using Pascal transform, Fundam. J. Math. Appl., (2019), 56-62.
- [5] Aydın S., Polat H., Some Pascal spaces of difference sequence spaces of order m , Conf. Proc. Sci. Technol., 2(1), (2019), 97–103.
- [6] Aydın C., Başar F., Some new paranormed sequence spaces, Inform. Sci., 160 (2004), 27–40.
- [7] Baliarsingh P., Some new difference sequence spaces of fractional order and their dual spaces, Appl. Math. Comput., 219 (2013), 9737-9742.
- [8] Baliarsingh P., Dutta S., A unifying approach to the difference operators and their applications, Bol. Soc. Parana. Mat., 33 (2015), 49-57.
- [9] Başar F., Summability Theory and its Applications, 2nd ed., CRC Press/Taylor and Francis Group, Boca Raton. London. New York, 2022.
- [10] Başar F., Altay B., On the space of sequences of p -bounded variation and related matrix mappings, (English, Ukrainian summary) Ukrain. Mat. Zh., 55(1) (2003), 108–118; reprinted in Ukrainian Math. J. 55(1) (2003), 136–147.
- [11] Dutta S., Jena D., Fractional difference sequence spaces via Pascal mean, International Conference of Mathematical Sciences and Applications, (2023), Communicated.

- [12] Et M., Çolak R., On generalized difference sequence spaces, *Soochow J. Math.*, 21 (1995), 377-386.
- [13] Grosse-Erdmann K. G., Matrix transformation between the sequence spaces of Maddox, *J. Math. Anal. Appl.*, 180 (1993), 223-238.
- [14] Jena D., Dutta S., Fractional ordered Euler Riesz difference sequence spaces, *Proyecciones*, 42(5) (2023), 1355-1372.
- [15] Jena D., Dutta S., Bell difference sequence spaces, *Jnanabha*, 54, 1 (2024), 53-61.
- [16] Kirişçi M. and Başar F., Some new sequence spaces derived by the domain of generalized difference matrix, *Computers and Mathematics with Applications*, 60(5) (2010), 1299-1309.
- [17] Kizmaz H., On certain sequence spaces, *Canad. Math. Bull.*, 24 (1981), 169-176.
- [18] Maddox I. J., Spaces of strongly summable sequences, *Quart. J. Math. Oxford*, 18(2) (1967), 345-355.
- [19] Malkowsky E., Başar F., A survey on some paranormed sequence spaces, *Filomat*, 31(4) (2017), 1099-1122.
- [20] Mursaleen M., Başar F., *Sequence Spaces: Topics in Modern Summability Theory*, CRC Press/Taylor and Francis Group, Series: Mathematics and Its Applications, Boca Raton. London. New York, 2020.
- [21] Polat H., Some new Pascal sequence spaces, *Fundam. J. Math. Appl.*, 1 (2018), 61-68.
- [22] Ye,silkayagil Savaşçı M., Başar F., Domain of the Cesàro mean of order α in Maddox's space ℓ_p , *Publ. Inst. Math. (Beograd) (N.S.)*, 114(128) (2023), 19-38.
- [23] Ye,silkayagil M., and Başar F., On the paranormed Nörlund sequence space of non-absolute type, *Abstr. Appl. Anal.*, 2014, Article ID 858704, 9 pages, 2014.